Extracting Individual Asset Return Volatility From An Index

Gary Schurman, MBE, CFA

November, 2009

Assume that we are in the business of insuring automobile lease residuals. When an automobile is leased, the lessee has the option to purchase the vehicle at the end of the lease at a set price (residual value). The purchase option price is set at the beginning of the lease. If at the end of the lease the vehicle's wholesale value is greater than the residual value, the lessee purchases the vehicle and realizes a gain equal to the difference between the vehicle's wholesale value and residual value. If at the end of the lease the vehicle's wholesale value is less than the residual value the lessee returns the vehicle to the lessor (usually a financial institution) and the lessor realizes a loss equal to the difference between the vehicle's residual value and wholesale value. The lessor has the worst of both worlds, vehicle's with embedded gains never get returned but vehicles with embedded losses do. The lessee has the option to *put the vehicle back* to the lessor. The lessor wants us, the insurer, to cover the lessor against losses when the put options are exercised.

We know from our college finance classes that options increase in value with volatility of the underlying asset. We know that the used car index has an annual volatility of approximately 6.00%. If vehicle values are not perfectly correlated, the volatility of the index is less than the volatility of the individual assets. Our task is to estimate the individual asset volatilities so that we can correctly price our insurance product.

Legend of Symbols

N	=	Number of assets in index
R	=	Annual percentage return on the index
V	=	Total dollar value of assets in index
W_i	=	Weight of asset i in index
r_i	=	Annual percentage return on asset i
v_i	=	Dollar value of asset i
z_i	=	Normally-distributed random number with mean = 0 and variance = 1
μ_i	=	Expected annual percentage return of asset i
ρ_{ij}	=	Correlation coefficient between asset i and j
σ_i	=	Expected annual percentage volatility of asset i. This is the unknown we are solving for

Constructing the Index

The annual return on asset i is the sum of the expected return and the unexpected return. We will assume that asset returns are normally-distributed. The equation for the annual return on any asset i is...

$$r_i = \mu_i + \sigma_i z_i \tag{1}$$

The annual portfolio percentage return is...

$$R = \sum_{i=1}^{N} v_i r_i \div V \tag{2}$$

If we define W_i to be $v_i \div V$ then equation (2) becomes...

$$R = \sum_{i=1}^{N} w_i r_i = \sum_{i=1}^{N} w_i(\mu_i + \sigma_i z_i) \tag{3}$$

Index First and Second Moments

Because portfolio return is the sum of expected and unexpected asset returns, there is a distribution around the portfolio return mean. The first moment of the return distribution is...

$$\begin{aligned}
\mathbb{E}\Big[R\Big] &= \mathbb{E}\Big[\sum_{i=1}^{N} w_i(\mu_i + \sigma_i z_i)\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^{N} w_i \mu_i\Big] + \mathbb{E}\Big[\sum_{i=1}^{N} w_i \sigma_i z_i\Big] \\
&= \sum_{i=1}^{N} w_i \mu_i
\end{aligned} \tag{4}$$

The second moment of the return distribution is...

$$\begin{aligned}
\mathbb{E}\Big[R^2\Big] &= \mathbb{E}\Big[\sum_{i=1}^{N} w_i(\mu_i + \sigma_i z_i) \sum_{j=1}^{N} w_j(\mu_j + \sigma_j z_j)\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j (\mu_i + \sigma_i z_i)(\mu_j + \sigma_j z_j)\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j (\mu_i \mu_j + \mu_i \sigma_j z_j + \mu_j \sigma_i z_i + \sigma_i \sigma_j z_i z_j)\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \mu_i \mu_j\Big] + \mathbb{E}\Big[\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j z_i z_j\Big] \\
&= \Big[\mathbb{E}\Big[r_p\Big]\Big]^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov_{ij}
\end{aligned} \tag{5}$$

Index Mean and Variance

The mean of the return distribution is equation (4), the first moment of the distribution. The variance of the return distribution is the second moment (equation (5)) minus the square of the first moment (equation (4)). The equation for portfolio return variance is...

$$\begin{aligned}
variance &= \Big[\mathbb{E}\Big[r_p\Big]\Big]^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov_{ij} - \Big[\mathbb{E}\Big[r_p\Big]\Big]^2 \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov_{ij}
\end{aligned} \tag{6}$$

Extract Individual Asset Variance From Index Variance

According to (6) portfolio variance is a weighted covariance matrix. We will redefine covariance between asset i and asset j as the product of the individual asset volatilities (standard deviations) and the correlation coefficient.

$$\begin{aligned}
\rho_{ij} &= \frac{cov_{ij}}{\sigma_i \sigma_j} \\
cov_{ij} &= \rho_{ij} \sigma_i \sigma_j
\end{aligned} \tag{7}$$

In order to extract individual asset variance we have to make some simplifying assumptions. We will assume equal asset weights, equal volatilities and equal correlation coefficients (when $i \neq j$) such that...

$$w_i = w_j = ... = \frac{1}{N} \tag{8}$$

$$\sigma_i = \sigma_j = ... = \sigma \tag{9}$$

$$\rho_{ij} = \rho_{jk} = ... = \rho \tag{10}$$

Using the above simplifying equations we can rewrite equation (6) as...

$$
\begin{aligned}
variance &= \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov_{ij} \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{1}{N}\frac{1}{N}\sigma_i\sigma_j\rho_{ij} \\
&= \frac{\sigma^2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij} \\
&= \frac{\sigma^2}{N^2}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}|i \neq j + \sum_{i=1}^{N}\sum_{j=1}^{N} 1|i = j\right] \\
&= \frac{\sigma^2}{N^2}\left[N(N-1)\rho_{ij} + N\right] \\
&= \frac{\sigma^2}{N}\left[(N-1)\rho_{ij} + 1\right] \\
&= \sigma^2\left[\frac{(N-1)}{N}\rho + \frac{1}{N}\right]
\end{aligned} \tag{11}
$$

Note: As N goes to infinity, variance goes to $\sigma^2\rho$.

Problem Solution

We can use equation (6) to solve our problem. Equation (11) requires that we define the following inputs...

Index volatility (std dev)	=	6.00%
Number of assets in index	=	100
Pair-wise correlation	=	0.25

Estimates of individual asset volatilities given number of assets in index...

N	**Volatility**
1	6.00%
5	9.49%
10	10.52%
20	11.19%
100	11.82%
1000	11.98%
10000	12.00%

Based on the table above and 100 assets in the index, individual annual return volatility is 11.82%. We will use this number as asset volatility when pricing our insurance product.